Logic & Theory of Meaning
Module A: Logic

CLASSICAL AND INTUITIONISTIC LOGIC

INTRODUCTION

1. Logic

Logic is often defined as the study of valid arguments.

An argument is what a person produces when he makes a statement and gives reasons for believing (or accepting as true) the statement. The statement itself is called the *conclusion* of the argument; reasons for believing the conclusion are called the *premises* of the argument. The process through which one arrives at accepting as true the conclusion on the basis of accepting as true the premises is called *inference*.

Let us consider, for example, a situation in which a teacher observes that John has not come to school; the teacher telephones John's mother and is informed that John is not sick; he infers that John skived off. The teacher's reasoning may be purely mental, or even unconscious; but if he were asked why he believes that John skived off, he would make his reasoning explicit more or less in the following form:

(1)
(a) If John is absent, then either he is sick or he skived off
(b) John is absent.
(c) John is not sick.

THEREFORE (d) John skived off.

Not only is (1) an argument: we have also a clear intuition that it is a logically *valid* argument. On the contrary, the following argument is clearly invalid:

(2)
(a) Every man is mortal.
(b) Socrates is mortal.

THEREFORE (c) Socrates is a man.

A preliminary, and fundamental, question is: what *is* a logically valid argument? We have more or less firm intuitions about which arguments are logically valid and which are not, but our question here is different: how should the the logical validity of an argument be defined or at least conceptually characterized?

The answer is not unique. As D. Prawitz remarks,[1]

> From this beginning [Plato's and Aristotle's writings] there have been at least three basic intuitions about what it is for an inference to be logically valid – or, as we also say, for its conclusion to *follow logically* from its premisses, or to *be a logical consequence* of its premisses. The most basic one, which goes almost without saying, is that a valid inference is truth-preserving: if the premisses are true, so is the conclusion. If the conditional here is understood as material implication, this means only that it is not the case that the premisses are true and the conclusion is false. Of course, the satisfaction of this condition is not enough to make the inference valid. Two further conditions, which occurr more or less explicitly in Aristotle's writings, must also be satisfied:
>
> > (1) It is because of the logical form of the sentences involved, and not because of their specific content, that the inference is truth-preserving.
> > (2) It is impossibile that the premisses are all true but the conclusion is false – or, in positive terms, it is necessary that if the premisses are true, then so is the conclusion.

[1]Prawitz (2005), p. 672.

There are two main answers to our question, according to how conditions (1)-(2) are conceptually articulated; and the two answers lead to two different paradigms of logical validity, and therefore to two different logics: classical logic and intuitionistic logic.

To begin with, some words about the formalization of logic.

2. Formalization of Logic

Typically, ordinary deductive reasoning takes place in a natural language, or perhaps a natural language augmented with some mathematical symbols. On the contrary, modern logic is formulated and developed in formal languages. So the question arises of the relationship between a natural language and a formal language. It may help to sketch several options on this matter.

One view is that the formal languages accurately exhibit actual features of certain fragments of a natural language. Some philosophers claim that declarative sentences of natural language have underlying *logical forms* and that these forms are displayed by formulas of a formal language. Other writers hold that (successful) declarative sentences express *propositions*; and formulas of formal languages somehow display the forms of these propositions. On views like this, the components of a logic provide the underlying deep structure of correct reasoning. A chunk of reasoning in natural language is correct if the forms underlying the sentences constitute a valid or deducible argument.

Another view, held at least in part by Gottlob Frege and Wilhelm Leibniz, is that because natural languages are fraught with vagueness and ambiguity, they should be *replaced* by formal languages. A similar view, held by W. V. O. Quine, is that a natural language should be *regimented*, cleaned up for serious scientific and metaphysical work. One desideratum of the enterprise is that the logical structures in the regimented language should be transparent. It should be easy to "read off" the logical properties of each sentence. A regimented language is similar to a formal language regarding, for example, the explicitly presented rigor of its syntax and its truth conditions. On a view like this, deducibility and validity represent *idealizations* of correct reasoning in natural language. A chunk of reasoning is correct to the extent that it corresponds to, or can be regimented by, a valid or deducible argument in a formal language.

We shall describe two logical systems:

– in propositional logic we can express statements as a whole, and combinations of them. Intuitively, a statement is a sentence in which something is told about some reality, and which can be true or false about that reality. For example, if A is the statement "Socrates is a man", and B means "Socrates is a dog", then $A \supset \neg B$ says: "if Socrates is a man, then Socrates is not a dog".

– in (first order) predicate logic a statement has a specific inner structure, consisting of terms and predicates. Terms denote objects in some reality, and predicates express properties of, or relations between, those ob- jects. For example, the same example as before might be expressed as $MAN(s) \supset \neg DOG(s)$. Here, MAN and DOG are predicates, and s is a term, all with obvious meanings. One further possibility on predicate logic is that we can speak about some and all objects. For example, $\forall x MAN(x)$ means that every x is a man.

I. CLASSICAL LOGIC

1. Validity and Logical Validity

As we have seen, a valid inference is truth-preserving; this justifies the following

Definition 1:
An argument is *valid* if and only if (henceforth iff) there is no possible situation in which its premises are all true and its conclusion is false.

Now, what is *logical* validity? The basic idea of the classical answer is that it is validity in virtue of the *logical form* of the sentences involved.

What is the *logical form* of a sentence? In order to answer we must have a clear idea of what is a *logical constant*. Here I shall simply give a list of the main logical constants in English: the binary *connectives* "and" (conjunction), "(either)...or" (disjunction), "if...then" (implication), "if and only if" (biimplication); the unary connective "not" (negation); the *quantifiers* "all" (universal quantifier) and "some" (existential quantifier); the *modal operators* "necessarily" and "possibly"; later we will introduce as a logical constant also the zero-ary connective ⊥ (the Absurd), the binary predicate = for identity, the modal connnectives "necessarily" (or "it is necessary that") and "possibily" (or "it is possibile that"). Consider for example the sentence (1)(a): it is an implication having two sub-sentences: the first (usually called *antecedent*) is logically simple or *atomic* (it does not contain logical constants), the second (usually called *consequent*) is logically complex, and it is a disjunction of two atomic sentences. We can put into evidence this logical form by substituting the atomic sentences with the variables p, q and r; the resulting form (or schema) is the following:

(3) If p, then either q or r.

Now we must explain the idea that an argument is valid *because of* the logical form of the sentences involved, and *not because of* their specific content. Essentially this means that the validity of the argument *depends* on the meaning of the logical constants occurring in its sentences, and does not depend on the meaning of the nonlogical constants occurring in its sentences. As a consequence, a logically valid argument is an argument such that every substitution of the nonlogical constant occurring in it results in in a valid argument. For example, consider (1): if we (uniformly) replace the atomic sentences occurring in it with others having different meaning (and structure), say with, respectively, "Tom speaks English", "Tom was born in England", "Tom learnt English at school": we obtain (with some adjustement):

(1')
(a) If Tom speaks English, then either he was born in England or he learnt English at school
(b) Tom speaks English
(c) Tom was not born in England
____,
THEREFORE (d) Tom learnt English at school

which is valid; and the same is true for every other substitution of the atomic sentences (i.e. of all the nonlogical expressions occurring in (1)) with other atomic sentences. This shows that the validity of (1) does not depend on the meanings of the nonlogical expressions occurring in it. Viceversa, the validity of (1) *does depend* on the meaning of the logical constants occurring in it:

for example, if we substitute “either...or” with “either not...or”, we obtain

(1")
(a) If John is absent, then eitherif John is not sick or John skived off
(b) John is absent.
(c) John is not sick.
———————————————————————————,
THEREFORE (d) John skived off.

which is not valid [why?[2]].

We could therefore define the (classical) logical validity of an argument in the following way:

Definition 2:
An argument is *logically valid* iff every substitution, in it, of the nonlogical expressions with other expressions of the same category results in a valid argument.

Here is an example of a valid but not logically valid argument:

(4)
(a) John is a bachelor
———————————————.
THEREFORE (b) John is not married

It is valid, since there is not a possible situation in which the premise is true and the conclusion is false (i.e. in which John is married). But it is not logically valid, because there is some nonlogical expression occurring in it such that, if we substitute it with another expression of the same category, we obtain an invalid argument; for example, let us substitute the predicate “is a bachelor” with the predicate “is young”: we obtain

(4')
(a) John is young
———————————————,
THEREFORE (b) John is not married

which is not valid.

Definition 2 (which essentially reflects Bolzano’s ideas about logical consequence), has a shortcoming. As we have seen, the intuitive idea informing it is that the validity of a logically valid argument does not depend on the meanings of the nonlogical constants, so that it remains valid however we vary such meanings. However, Definition 2 does not make reference to variations of meaning, but to substitutions of nonlogical expression with other expressions; the problem is that our language may contain less nonlogical expressions to substitute to a given one than possible meanings of the given expression: when this is the case, it can happen that an argument is valid according to definition 2 but intuitively invalid. For example, (4) is intuitively logically invalid, but if our language contained only the predicate “is a bachelor”, there would no substitution of

[2] In order to establish that it is not valid we must reason. The question is: is there a possible situation in which all the premises are true and the conclusion is false? Consider an arbitrary situation in which all the premises are true; since (a) and (b) are true, so is (*):
(*) Either John is not sick, or John skived off;
since (c) is true, (*) may be true althoug “John skived off” is false; so *there is* a situation in which (a)-(c) are true and (d) is false.

nonlogical expressions resulting in an invalid argument. It is therefore better to make reference directly to variations of meaning of the nonlogical expressions, i.e. to the totality of *interpretations* of the given language.[3] We arrive in this way to the following

Definition 3:
An argument is *logically valid* iff, for every model $\mathcal{M}$, it is valid in $\mathcal{M}$.[4]

This is how Tarski conceptually articulated conditions (1)-(2) mentioned by Prawitz. [5]

Validity and logical validity can be ascribed also to sentences. The validity of a sentence can be seen as a particular case of the validity of an argument, since a sentence can be seen as a particular case of an argument: the case of an argument with no premises (or: with the empty set of premises).

---

[3] An *interpretation*, or *model*, of a language is something that provides a meaning to the non-logical expressions of the language, and thereby provides information to determine the truth values of alla the formulas of the language. A more precise definition ios possible only when a language is actually given.

[4] Given Definition 1, this is equivalent to say that an argument is logically valid iff, for every model $\mathcal{M}$, if its premises are all true in $\mathcal{M}$, then its conclusion is true in $\mathcal{M}$.

[5] In Tarski (1935).

2. Classical Propositional Logic

2.1. Language **2.2 Semantics**	}**Insert J. MacFarlane, "Propositional Logic Review" (only pp. 1-5)** online: Macfarlane propositional-logic-notes.pdf

2.3. Proofs

A *proof system* is intended to permit us to *recognize* that an argument is valid, when it is. A proof system permitting us, moreover, to recognize that an argument is *not* valid, when it is not, is called a *decision system*.

There are several proof systems for propositional logic, and several decision systems. We now describe an extremely elegant and efficient proof system for propositional logic which is also a decision system: the system of *tableaux*.

Tableaux are essentially a method for testing the consistency of a set of formulas;[6] but they can be used for testing the validity of arguments as well, owing to the fact that the validity of an argument can be reduced to the consistency of a set of formulas. Given an argument with premises $p_1,..., p_n$ and conclusion q, its *counterexample set* is the set $\{p_1,..., p_n, \neg q\}$. From Definition 1 it follows that

(5) An argument is valid iff its counterexample set is inconsistent.

So the *tableau for an argument* can be seen as the systematic attempt to describe a *counterexample* to the argument, i.e. a situation in which all the sentences of the counterexample set are true: if the attempt succeeds, the argument is invalid; if the attempt does not succeed, in the sense that it runs into a contradiction, the argument is valid.

Before we state the rules for the construction of tableaux, we shall illustrate the construction with an example. Consider argument (1), which can now be formalized in the following way:

(6)
$$A\supset(B\vee C)$$
$$A$$
$$\neg B$$
$$\overline{\qquad\qquad}$$
$$\therefore\ C$$

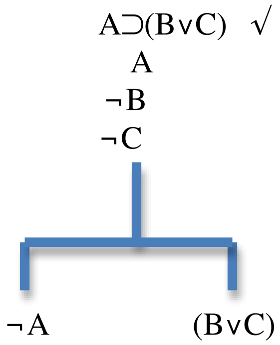
Testing whether it is valid amounts to test whether its counterexample set is consistent. So let us start by writing the sentences of the counterexample set one below the other:

(7)
$$A\supset(B\vee C)$$
$$A$$
$$\neg B$$
$$\neg C$$

This is the first step. Do the sentences in (7) describe a situation in which all the sentences of the counterexample set are true? This is not clear, because the first sentence is logically complex, and it is not immediately evident how a situation should be for it to be true. But we can try to describe a situation in which it is true *by means of sentences shorter than it*, and continue in this way till we reach a description made up of sentences so short that it is immediately clear whether it describes a situation in which all the sentences of the counterexample set are true. So, let us consider the sentence $A\supset(B\vee C)$: it is an implication; looking at the truth-table for implication we see that it is true in two cases: when the antecedent is false, or when the consequent is true; let us register this alternative in the following way:

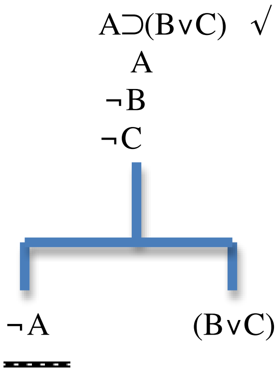
[6] A set of formulas is *consistent* iff there is a possible situation in which all its formulas are true; *inconsistent* otherwise.

(8)



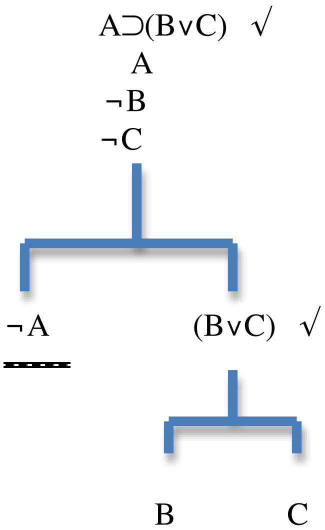
(The tick at the right of the first sentence means that the sentence has been dealt with.) Consider the first (leftmost) alternative: it is the description of a situation in which A is true (since A occurs in the root of the tableau) and ¬A is true (since ¬A occurs in the left branch), i.e. A is false (look at the truth-table for ¬); this is a contradiction: a sentencence cannot be be both true and false. So, the first case leads to a contradiction: it is an unsuccessful attempt at describing a counterexample. We take note of this by barring or *closing* the branch:

(9)



No other sentences of the root require to be dealt with (the negation of an atomic sentence is logically complex, but its truth amounts to the falsity of the atomic sentence). So the first (and only) sentence to consider is the one describing the second alternative; we have a complex sentence; applying the same idea as before, we observe that B∨C is true in two alternative cases, which we register in the following way:

(10)



Now we observe that in both cases we reach a contradiction, and we register this fact:

(11) A⊃(B∨C) √

A

¬B

¬C

¬A   (B∨C) √

B   C

------   ------

No other sentences require to be dealt with; so the tableau is *completed*. This tableau is also *closed*, since all its branches are closed. (A tableau some of whose branches are not closed is *open*.) This means that all possible ways to describe a counterexample to the argument led to a contradiction, so the argument is valid.

Let us consider the case of an invalid argument: argument (1”). The root of the tableau for it is the following:

(12) A⊃(¬B∨C)

A

¬B

¬C

The second step is similar to the second step of the preceding example:

(13) A⊃(¬B∨C) √

A

¬B

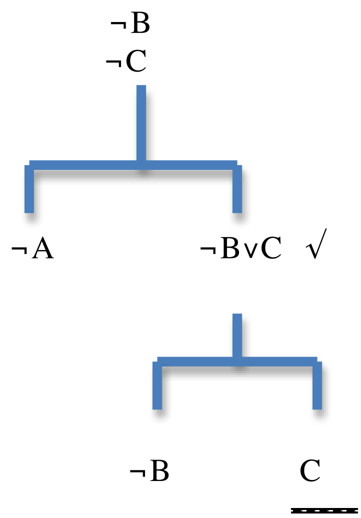
¬C

¬A   ¬B∨C

Now the first (and only) sentence to consider is ¬B∨C, which is true in the two alternative cases described below:

(14) A⊃(B⊃C) √

A



The rightmost branch closes, owing to the presence in it of ¬C; but the branch terminating with ¬B is open: the tableau is therefore open; moreover, the open branch describes a counterexample to the argument: a situation in which A is true, and B and C are false.

Till now we have spoken of tableaux as a method for testing consistency and validity, two notions we have defined as *semantic* notions, making essential use of the notion of truth. But we want to describe the system of tableaux as a *proof* system, and the notion of proof is generally understood as a *syntactic* notion, not involving the notion of truth. This is possible, if we conceive tableaux as constructed according to rules that do not mention truth (although they reflect of course our intuitions about the truth-conditions of the sentences analyzed). Here are the rules:

Rules for the construction of tableaux.
We now state all the rules for propositional logic in schematic form; explanations immediately follow.

(15) (a) ¬¬p

p

(b) p∧q ¬(p∧q)

p ¬p ¬q
q

(c) p∨q ¬(p∨q)

p q ¬p
¬q

(d) p⊃q ¬(p⊃q)

¬p q p
¬q

Explanations.

Rule (a) means that from ¬¬p one can directly infer p, in the sense that one can subjoin p to any branch passing through ¬¬p. Rule (b) means that p∧q directly yields both p, q, whereas ¬(p∧q) branches into ¬p, ¬q. Rules (c) and (d) can be understood analogously.

Since formulas can be seen as particular cases of arguments (arguments with zero premises), tableaux can be constructed also for formulas: a *tableau for A* is a tableau having as a root the negation of A. Here is an example, which illustrates also other aspects of the construction of a tableau. We wish to test the formula (A∨(B∧C))⊃((A∨B)∧(A∨C)); the tableau is the following; the explanation is given after the tableau:

(16)

(1)	¬((A∨(B∧C))⊃((A∨B)∧(A∨C)))	√
(2)	A∨(B∧C)	√
(3)	¬((A∨B)∧(A∨C))	√

(4) A	(5) B∧C √
	(6) B
	(7) C

(8) ¬(A∨B)√	(9) ¬(A∨C)√	(10) ¬(A∨B)√	(11) ¬(A∨C)√
(12) ¬A	(14) ¬A	(16) ¬A	(18) ¬A
(13) ¬B	(15) ¬C	(17) ¬B	(19) ¬C

The tableau was contructed as follows. Line (1) is the formula to test preceded by negation. We analyze it: it is a negated implication, so we apply to it the second rule of (15)(d), and we obtain its *direct* consequences (2) and (3), which we write immediately below it. (We put a tick at the right of it.) Now let us look at line (2): it is a disjunction, whose rule (the first of (15)(c)) is a *branching* rule: we apply it, obtaining the two branches (4) and (5). (We put a tick at the right of (2).) Line (5) is a conjunction,which, according to the conjunction rule, has two direct consequences: we write them below (5), as (6) and (7). (We put a tick at the right of (5).) Now look at line (3): it is the negation of a conjunction, whose rule is a branching rule; we also know that there are the two branches (4) and (5) (which intuitively describe two alternative attempts of arriving at a counterexample); so we apply the rule to *each* of the branches, obtaining the four branches (8), (9) and (10), (11).[7] (We put a tick at the right of (3).) Then we analyze the formulas of each of these branches, obtaining, as their direct consequences, the formulas (12)-(19). (We put ticks at the right of (8)-(11).)

When a tableau is constructed in the manner illustrated, a formula that has been analyzed need never be analyzed again (this is the meaning of ticks); as a consequence, after a finite number of steps we must reach a point where every formula has been analyzed (except atomic formulas and negations of atomic formulas). At this point the tableau is *complete*. This is the case for our tableau (16).

Now look at the leftmost branch of (16): it contains (12), i.e. ¬A, and (4), i.e. A: a contradiction; so we bar, or close, it with a line. Analogously we bar all the other branches. Since all the branches are closed, the tableau is *closed*.

[7] Notice that we could have analyzed formula (3) before formula (5). The reason of our choice is that it is more convenient; it is based on the idea of giving priority to formulas whose analysis requires *non-branching* rules); in this way, one will omit repeating the same formula on different branches, because one will have only one occurrence of the formula *above* all the branch points. For example, if we had analyzed (3) before (5), formulas (4) and (5), together with their consequences, should have been repeated on two different branches.

Finally, we say that an argument with premises in Γ and conclusion p is *syntactically valid* (in symbols $\Gamma \vdash_{CPL} p$) iff there is a closed tableau for it. In the particular case in which Γ is empty we say that p is a *theorem* (in symbols $\vdash_{CPL} p$), and the closed tableau for ¬p is a *proof* of p.

2.4. The relation of semantics and proofs.

The two following two fundamental metatheorems answer the question of how the definition of validity of an argument can be paired with a criterion of validity, i.e. with a way to recognize that an argument is valid:

(i) **Soundness (meta-)theorem**: If $\Gamma \vdash_{CPL} p$, then $\Gamma \vDash_{CPL} p$

(ii) **Completeness (meta-)theorem**: If $\Gamma \vDash_{CPL} p$, then $\Gamma \vdash_{CPL} p$.

3. Classical Predicate Logic

3.1. Language

The language of (first order) predicate logic contains:
- *individual constants* a,b,..., possibly with numerical subscripts;
- *individual variables* x, y,..., possibly with numerical subscripts;
- *predicates P, Q,...*, possibly with a numerical subscript; [8]
- the zero-place connective ⊥ (to be read as "The Contradiction" or "The False");
- the two-place connectives ∧, ∨, ⊃ (to be read as "and", "or", "if...then", respectively);
- the quantifiers ∀ (universal) and ∃ (existential) (to be read as "for all" and "some", respectively);

A *term* is an individual variable or an individual constant

Formulas are defined inductively by the clauses:
1. ⊥ is a formula;
2. If P is an n-ary predicate and $t_1,..., t_n$ are terms, then $P(t_1,...,t_n)$ is an (*atomic*) formula;
3. If A and B are formulas, A∧B , A∨B and A⊃B are formulas;
4. If A is a formula, ∀xA and ∃xA are formulas;
5. Nothing else is a formula.

Note: We shall often speak about the negation ¬A of A; but ¬ need not be be considered as a primitive symbol of our language, since negation can be defined: $\neg A =_{def} A \supset \bot$.

The *scope* of a quantifier is the formula directly following the quantifier; for example,
- in ∀x(Px⊃Qx), the scope of the quantifier is the formula (Px⊃Qx);
- in ∀xPx⊃Qx, the scope of the quantifier is the formula Px;
- in ∀x¬Px∨Qa, the scope of the quantifier is the formula ¬Px.

An occurrence of a variable x in a formula A is *bound* or *free* according as the occurrence belongs or does not belong to the scope of a quantifier that is immediately followed by x. For example, in ∀x(Py⊃Rxy)⊃Ryx there are three occurrences of x: the first is the one immediately following the quantifier (and simply individuates the variable); the second is bound, the third is free. A formula containing free occurrences of some variable is called an *open formula*; a formula without free occurrences of any variable is called a *closed formula* or *sentence*.

3.2 Semantics	**Insert:** **J. MacFarlane, "Predicate Logic Review" (only pp. 1-5)** online: Macfarlane predicate-logic-notes.pdf

3.3. Proofs

3.3.1 Substitution instances.

[8] Predicates can be classified as one-place, two-place, and in general *n*-place, depending on how many argument places they have. A logically perspicuous language would mark this, say, with a numerical superscript, but we will normally just leave it implicit.

It will be necessary to make reference to the uniform *substitution* of an expression occurring in a formula with another expression. We shall use the following notation. If A is a formula and t is a term, A[t/x] is the result of the substitution, in A, of every free occurrence of x with t; A[t/x] is called a *substitution instance* of A. Thus, for example, if A is the formula $P(x,b,x)$, then $A[a/x]$ is the formula $P(a,b,a)$; if A is the formula $\exists xP(x,b,x)$, then $A[a/x]$ is the formula $\exists xP(x,b,x)$; if A is the formula $P(y,b,x)$, then $A[a/x]$ is the formula $P(y,b,a)$.

3.3.2 Tableau Rules for Quantifiers.

The proof system for predicate logic is obtained by adding to the propositional rules for tableaux the following rules for quantified statements:

(17)

(a)

$\forall xA$	$\neg\forall xA$	
		(*)
$A(c/x)$	$\neg A(c/x)$	

(b)

$\exists xA$		$\neg\exists xA$
	(*)	
$A(c/x)$		$\neg A(c/x)$

(*) Provided c does not occur in the branch to which $\neg\forall xA$ (resp. $\exists xA$) belongs.

Let us explain the reasons of the proviso (*). Suppose that in a branch of a tableau we have examined the statement ∃xPx, obtaining as a result Pa; and that, subsequently, we find on the same branch another existential statement, for example ∃xQx. If we applied the rule to this statement disregarding the proviso, we would obtain Qa. As a consequence we would have on the same branch both Pa and Qa, which intuively means that *the same* individual a is P and is Q: a consequence that is not legitimated by the two statements ∃xPx and ∃xQx: from the fact that someone is old and someone is young, we cannot infer that someone is old and young.

The definition of syntactical validity is formally identical to the one for propositional logic. An argument with premises in Γ and conclusion p is *syntactically valid* (in symbols $\Gamma\vdash_{CPrL}p$) iff there is a closed tableau for it. In the particular case in which Γ is empty we say that p is a *theorem* (in symbols $\vdash_{CPrL}p$), and the closed tableau for ¬p is a *proof* of p.

3.4. The relation of semantics and proofs.

(i) **Soundness (meta-)theorem**: If $\Gamma\vdash_{CPrL}p$, then $\Gamma\vDash_{CPrL}p$

(ii) **Completeness (meta-)theorem**: If $\Gamma\vDash_{CPrL}p$, then $\Gamma\vdash_{CPrL}p$.

4. Classical Propositional Modal Logic

Insert J. MacFarlane, "Propositional Modal Logic" (only pp. 1-6)
online: MacFarlane modal-propositional-logic.pdf

II. INTUITIONISTIC LOGIC

Simmetry with the first part would require that we start with an explanation of how intuitionists conceive validity and logical validity. However, for reasons that will become clear below, in this second part it is better to reverse the order of exposition, starting with an illustration of intuitionistic formalized systems.

1. Intuitionistic Propositional Logic

1.1. Language

The language of Intuitionistic Propositional Logic IPL is the same as the language of Classical Propositional Logic, with one difference: we will introduce the new logical constant $\bot$ (to be intuitively understood as a constant for the False, or the Contradiction), and we will consider negation not as a primitive symbol, but as defined, in the following way:

$\neg p =_{def} p \supset \bot$

1.2 Semantics

Models

A *Kripke model for IPL* is a triple $\mathcal{K}$=<W,R,V>, where

- W is a nonempty set of objects (worlds, to be intuitively understood as knowledge states);
- R is a partial order (i.e. a reflexive, antisymmetrical and transitive relation) on W;
- V is a function (the valuation function), assigning a truth value to each pair of a propositional constant and a word (in symbols: $V(p,w) \in \{0,1\}$), and satisfying the following *monotonicity condition*:
  (*) If V(p,w)=1 and wRw', then V(p,w')=1.

Truth in a model and validity

Now, given an arbitrary Kripke model $\mathcal{K}$=<W,R,V> and a world $w \in W$, we define the notion "p is verified at w in $\mathcal{K}$" (in symbols: $\vDash_w^{\mathcal{K}} p$, or $\vDash_w p$ for short), by induction on the complexity of p:

- If p is a propositional constant, $\vDash_w p$ iff V(p,w)=1
- $\nvDash_w \bot$
- $\vDash_w q \wedge r$ iff $\vDash_w q$ and $\vDash_w r$
- $\vDash_w q \vee r$ iff $\vDash_w q$ or $\vDash_w r$
- $\vDash_w q \supset r$ iff, for every w', if wRw', then either $\nvDash_{w'} q$ or $\vDash_{w'} r$

As a consequence of our definition of negation and of the clause about $\bot$, we have that

- $\vDash_w \neg p$ iff, for every w', if wRw', then $\nvDash_{w'} p$.

Now we can define the notion "p is true in the model $\mathcal{K}$" (in symbols: $\vDash^{\mathcal{K}} p$):

$\vDash_w^{\mathcal{K}} p$ iff, for every $w \in W$, $\vDash_w^{\mathcal{K}} p$.

We will say that p is *false* in the model $\mathcal{K}$ (in symbols: $\nvDash^{\mathcal{K}} p$) iff p is not true in $\mathcal{K}$, i.e. iff, for some w∈W, $\not\models_w^{\mathcal{K}}$p. A model in which p is false is sometimes called a *countermodel* to p.

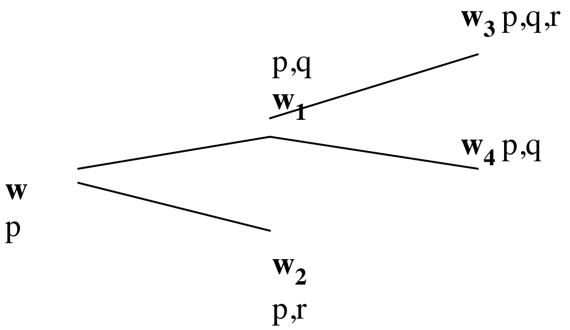
At this point we can define validity:

An argument (schema) with premisses in Γ and conclusion p is *IPL-valid* (in symbols: $\Gamma\models_{IPL}$p) iff there is not a Kripke model in which all the formulas in Γ are true and p is false.

As usual, the validity of a formula is a special case of the validity of an argument: the case in which Γ is empty. Therefore, p is *IPL-valid* (in symbols: $\models_{IPL}$p) iff there is not a countermodel to p.

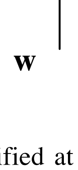
Intuitive interpretation.
The intuitive idea behind Kripke models is that sentences are not – for the intuitionist – absolutely true or false, but verified or not-verified relatively to a knowledge state, i.e. a state in which certain atomic sentences have been verified by a knowing subject.[9] Knowledge states can be conceived as points of an epistemic space; given a state w, one 'accesses' another state w' (which will be called *accessible* from w) by verifying new atomic sentences; the monotonicity condition (*) can be understood as the condition that the knowing subject does not forget what has been verified.

Let us suppose we are in state w, in which only p is verified; there are several possibilities concerning the verification of new sentences; here is one example:[10]



In this model only five knowledge states are possible. From w we can access either w_1, where we verify q, or w_2, where we verify r (in both cases 'remembering' that p has been verified at w, as (*) requires). From w_2 we cannot access other knowledge states, while from w_1 we can access either w_3 or w_4: in w_3 we verify r, in w_4 we acquire no new information; but this does not mean that the state w_4 is exactly like w_1: in w_1 the possibility to access w_3 is still open, while in w_4 we know enough to rule out the possibility that in the future r is verified.[11]

Let us conclude by presenting some countermodels to classical tautologies (verify that, in each case, the formula is false at w).
Countermodel to p∨¬p:

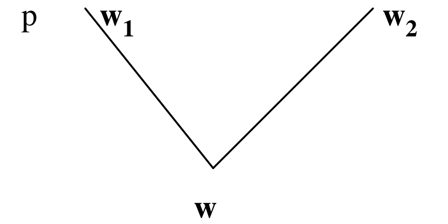


[9] A sentence p such that V(p,w)=1 is verified at w; a sentence p such that V(p,w)=0 is not-verified at w. If p is not verified at w, it can be either verified at a subsequent state or stay not-verified. If p is verified at w, condition (*) requires that it is verified at every state accessible from w.

[10] Writing "p" near the name of a world w amounts to stipulating that V(p,w)=1; writing nothing amounts to stipulating that V(p,w)=0.

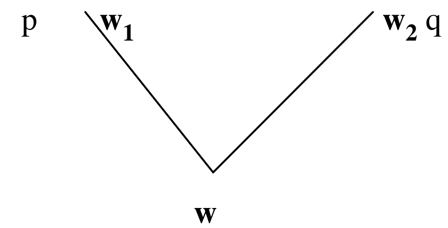
[11] A knowledge state is therefore individuated not only by information available in it, but also by the (possibily empty) class of states accessible from it.

Countermodel to ¬p∨¬¬p:

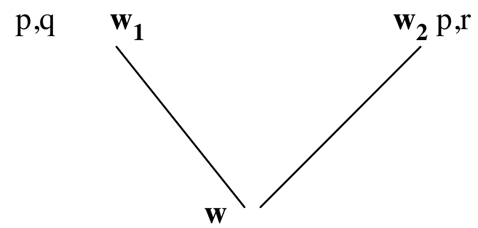


[Verify that the same model is a countermodel to (¬¬p⊃p)⊃(p∨¬p).]

Countermodel to (p⊃q)∨(q⊃p):



Countermodel to (p⊃(q∨r))⊃((p⊃q)∨(p⊃r)):



1.3. Proofs

The most natural proof system for intuitionistic logic is Gerhard Gentzen's Natural Deduction Gentzen's original aim in Gentzen (1935) was to give a formalization of logic as close as possible to the actual practice of mathematicians. (We shall see that natural deduction can serve as a proof system also for classical logic, but at the cost of some stretching.) To this purpose he analyzed each argument into atomic steps and stated the rules of such steps. His analysis was of a highly systematic character in that to each logical constant C two kinds of inference rules are associated: the former establishes the premisses under which we are entitled to infer a formula having C as its principal operator, thereby introducing it into an argument; the latter determines the conclusion we are authorized to infer from a formula having C as its principal operator (maybe together with other premisses), thereby eliminating it from an argument. Rules of the former kind will therefore be called *C-introductions*, and rules of the latter *C-eliminations*. Here are the rules for propositional logic:

Inference rules

∧I: $\dfrac{p \quad q}{p\wedge q}$ **∧E$_1$**: $\dfrac{p\wedge q}{p}$ **∧E$_2$**: $\dfrac{p\wedge q}{q}$

p q p∨q [p] [q]
  r r

$\vee\mathbf{I_1}$: $\dfrac{\quad}{p\vee q}$ $\qquad$ $\vee\mathbf{I_2}$: $\dfrac{\quad}{p\vee q}$ $\qquad$ $\vee\mathbf{E}$: $\dfrac{\qquad\qquad\qquad}{r}$

$\supset\mathbf{I}$: $\dfrac{\begin{matrix}[p]\\ q\end{matrix}}{p\supset q}$ $\qquad$ $\supset\mathbf{E}$: $\dfrac{p \quad p\supset q}{q}$

As concerns negation, remember that we have assumed that our language contains the constant ⊥ for 'the contradiction' or 'the false'; as a consequence ¬P is an abbreviation of P⊃⊥. Therefore the rules Gentzen gives for ¬:

$\neg\mathbf{I}$: $\dfrac{\begin{matrix}[p]\\ \bot\end{matrix}}{\neg p}$ $\qquad$ $\neg\mathbf{E}$: $\dfrac{p \qquad \neg p}{\bot}$

become particular cases of ⊃I and ⊃E, respectively.

To the preceding rules Gentzen adds

$(\bot_{\mathbf{Int}})$ $\dfrac{\bot}{p}$

which, he says, «occupies a special place among the schemata: it does not belong to a logical symbol, but to the propositional symbol ⊥.»[12]

Let us briefly explain how these rules operate in the construction of *derivations*. Consider the following derivation:

(18)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A]^1}{A\vee\neg A}\,\vee\mathbf{I} \qquad [\neg(A\vee\neg A)]^2}{\bot}\,\supset\mathbf{E}}{\neg A}\,\supset\mathbf{I}, 1}{A\vee\neg A}\,\vee\mathbf{I} \qquad [\neg(A\vee\neg A)]^2}{\bot}\,\supset\mathbf{E}}{\neg\neg(A\vee\neg A)}\,\supset\mathbf{I}, 2$$

At each step we may introduce an assumption, and assumptions are numerated. For example, at the first step we have entered assumption A, and we have assigned it the numeral 1 as superscript (we

[12] Gentzen (1935), p. 81. Notice that above we have characterized ⊥ as a zero-place connective, hence as a *logical* symbol. Gentzen's view may have been motivated by the observation that, unlike the logical symbols, ⊥ has not an introduction rule.

have not yet written the square brackets). After having introduced an assumption, we may either infer a formula from it according to some inference rule (at step two we have inferred $A\vee\neg A$ applying $\vee$**I**, entering on the right of the transition line the name of the rule), or introduce a new assumption (at step three where we introduced assumption 2). Of course, what is later inferred will *depend* on that assumption, until it is *discharged* by the application of some rule.[13] In this example assumption 1 is discharged at step five: we indicate the discharge of assumption 1 by putting it into square brackets and writing its numeral on the right of the transition line, after the name of the rule that has been applied.

In this derivation the formula $\neg(A\vee\neg A)$ is introduced twice as assumption; have we introduced one assumption or two? Two. However, we have assigned them the same numeral to indicate that we have discharged them at once. Such a simultaneous discharge was possible only because the two assumptions are occurrences of the same formula: assumptions of different forms cannot be discharged at once. But the simultaneous discharge of assumptions for the same form is not mandatory: in some cases it is convenient to discharge them at different steps. Consider for instance the following derivation:

$$(19)\qquad \dfrac{\dfrac{\dfrac{[A]^1 \qquad [A]^2}{A\wedge A}\,\wedge\mathbf{I}}{A\supset(A\wedge A)}\,\supset\mathbf{I},1}{A\supset(A\supset(A\wedge A))}\,\supset\mathbf{I},2$$

Since the two assumptions are discharged at different steps, they have been given two different numerals.

In general, dischargement of assumptions is to be considered as a right, not an obligation.

Consider the following proof:

$$(20)\qquad \dfrac{[A]^1}{A\supset A}\,\supset\mathbf{I},1;$$

has the rule $\supset$**I** been correctly applied? Yes: the premiss of the rule $\supset$**I** is that we have a derivation of B from the assumption A; but, as the rule for assumptions says, the single formula A *is* a derivation of A from the assumption A; consequently, it is perfectly correct to infer $A\supset A$ discharging the assumption A.

Consider now the proof:

$$(21)\qquad \dfrac{\dfrac{[A]^1}{B\supset A}\,\supset\mathbf{I}}{A\supset(B\supset A)}\,\supset\mathbf{I},1;$$

the first application of $\supset$**I** is correct for another reason. The premiss of the rule $\supset$**I** is that we have a derivation of B from the assumption A, but not from A alone: B may be seen as depending on any

[13] The only rules permitting discharge of assumptions are $\vee$**E** and $\supset$**I**.

other assumptions, since, if B can be inferred from A, it can be inferred from A together with any other assumption.

If the justifications of the correctness of (21) strikes you as unnatural, notice that the same conclusion can be obtained by the following (longer but) undoubtedly correct proof:

(22)
$$\cfrac{\cfrac{\cfrac{\cfrac{[A]^1 \quad [B]^2}{A\wedge B}\,\wedge\mathbf{I}}{A}\,\wedge\mathbf{E}}{B\supset A}\,\supset\mathbf{I},2}{A\supset(B\supset A)}\,\supset\mathbf{I},1;$$

We are now ready to give a more rigorous definition of *derivation*, *open assumption*, and *discharged* (or eliminated) *assumption*.

Basis
The single-node tree constituted by a single occurrence of p is a derivation of p from the open assumption p.

Inductive step
If Π_1, Π_2 and Π_3 are derivations, a new derivation is a tree constructed according to the inference rules specified above. More precisely, Π is a derivation if it has one of the following forms:

$(\bot_{\mathbf{Int}})$
$$\frac{\begin{matrix}\Pi_1\\ \bot\end{matrix}}{p}$$

$\wedge\mathbf{I}$:
$$\frac{\begin{matrix}\Pi_1 & \Pi_2\\ p & q\end{matrix}}{p\wedge q}$$

$\wedge\mathbf{E_1}$:
$$\frac{\begin{matrix}\Pi_1\\ p\wedge q\end{matrix}}{p}$$

$\wedge\mathbf{E_2}$:
$$\frac{\begin{matrix}\Pi_1\\ p\wedge q\end{matrix}}{q}$$

$\vee\mathbf{I_1}$:
$$\frac{\begin{matrix}\Pi_1\\ p\end{matrix}}{p\vee q}$$

$\vee\mathbf{I_2}$:
$$\frac{\begin{matrix}\Pi_2\\ q\end{matrix}}{p\vee q}$$

$\vee\mathbf{E}$:
$$\frac{\begin{matrix} & [p] & [q]\\ \Pi_1 & \Pi_2 & \Pi_3\\ p\vee q & r & r\end{matrix}}{r}$$

$\supset\mathbf{I}$:
$$\frac{\begin{matrix}[p]\\ \Pi_1\\ q\end{matrix}}{p\supset q}$$

$\supset\mathbf{E}$:
$$\frac{\begin{matrix}\Pi_1 & \Pi_2\\ p & p\supset q\end{matrix}}{q}$$

Applications of $\vee$**E** and of $\supset$**I** discharge all open assumptions of the forms indicated between square brackets. Assumptions that are not discharged are open.

We will use the symbol "$\Gamma \vdash_{IPL} p$" to say that there is a derivation of p from the (open) assumptions in Γ; when the set Γ is empty (i.e., all assumptions from which p has been derived are discharged), we write $\vdash_{IPL} p$, and we say that p is a(n IPL) *theorem*.

1.4. The relation of semantics and proofs.

As in the case of classical propositional logic CPL, we can ask questions about how Kripke semantics and Gentzen natural deduction system are related. In fact, the following facts can be proved:

(i) **Soundness (meta-)theorem**: If $\Gamma \vdash_{IPL} p$, then $\Gamma \vDash_{IPL} p$

(ii) **Completeness (meta-)theorem**: If $\Gamma \vDash_{IPL} p$, then $\Gamma \vdash_{IPL} p$

2. Intuitionistic Predicate Logic IPrL

2.1. Language

The language of Intuitionistic Predicate Logic IPrL is the same as the language of Classical Predicate Logic CPrL, with $\bot$ as primitive and negation defined as indicated above.

2.2 Semantics

Models

A *Kripke model for IPrL* is a quadruple $\mathcal{K}$=<W,R,D,⟦⟧>, where

- W is a nonempty set of objects (worlds, to be intuitively understood as knowledge states);
- R is a partial order (i.e. a reflexive, antisymmetrical and transitive relation) on W;
- D is a function (the domain function) associating to each world w a non-empty set D_w of individuals (the domain of w) and such that, if wRw′, then $D_w \subseteq D_{w'}$;
- ⟦⟧ is a function (the interpretation or denotation function) assigning a denotation, to each pair consisting of an individual constant, or a predicate letter, and a world. More specifically, if c is an individual constant, then $[\![c]\!]_w \in D_w$; if P is an n-ary predicate, then $[\![P]\!]_w \subseteq D_w{}^n$. The following monotonicity constraint is imposed onto I:
  (*) If wRw′, then $[\![P]\!]_w \subseteq [\![P]\!]_{w'}$

Truth in a model and validity

An *assignment* is a function g assigning, to each variable v, an element of $\bigcup_{w \in W} D_w$.

Given an arbitrary Kripke model $\mathcal{K}$=<W,R,D,I>, a world w∈W, an assignment g and, and a term t, the *semantic value of t relative to w and g* (in symbols: $[\![t]\!]_{w,g}$) is defined as follows:
- if t is a variable v, then $[\![t]\!]_{w,g} = g(v)$
- if t is a constant c, then $[\![t]\!]_{w,g} = [\![c]\!]_w$

Now, given an arbitrary Kripke model $\mathcal{K}$=<W,R,D,I> and a world w∈W, we define the notion "A is verified at w in $\mathcal{K}$ relative to the assignment g (in symbols: $\vDash_{w,g}{}^{\mathcal{K}}A$, or $\vDash_{w,g}A$ for short), by induction on the complexity of the (open or closed) formula A:

- $\vDash_{w,g}Pt_1,..., t_n$ iff $<[\![t_1]\!]_{w,g},..., [\![t_n]\!]_{w,g}> \in [\![P]\!]_w$
- $\nvDash_{w,g}\bot$
- $\vDash_{w,g}A \wedge B$ iff $\vDash_{w,g}A$ and $\vDash_{w,g}B$
- $\vDash_{w,g}A \vee B$ iff $\vDash_{w,g}A$ or $\vDash_{w,g}B$
- $\vDash_{w,g}A \supset B$ iff, for every w′, if wRw′, then either $\nvDash_{w',g}A$ or $\vDash_{w',g}B$
- $\vDash_{w,g}\forall vB$ iff, for every $g' =_v g$ and for every w′, if wRw′, then $\vDash_{w',g'}B$
- $\vDash_{w,g}\exists vB$ iff, for some $g' =_v g$, $\vDash_{w,g'}B$

If A is a sentence (i.e., a closed formula), A is verified at w in $\mathcal{K}$ ($\vDash_w A$ for short) iff, for all assignments g, $\vDash_{w,g}A$. The definitions of truth in a model and of IPrL-validity are formally identical to the corrsponding definitions for IPL.

2.3. Proofs

Natural Deduction Rules for Quantifiers

$$\forall\mathbf{I}:\quad \frac{A(y/x)}{\forall xA}(*) \qquad \forall E:\quad \frac{\forall xA}{A(t/x)}$$

$$\exists\mathbf{I}:\quad \frac{A(t/x)}{\exists xA} \qquad \exists\mathbf{E}:\quad \frac{\exists xA \qquad \begin{matrix}[A(y/x)]\\ B\end{matrix}}{B}(**)$$

(*) Provided y does not occur in any of the assumptions that A(y/x) depends on, nor in ∀xA.
(**) Provided y does not occur in ∃xA, nor in B, nor in any assumption that B depends on, except A(y/x).

Let us explain the reasons of the provisos (*) and (**).
- Proviso (*). The rule **∀I** is justified by the meaning of the universal quantifier, according to which we are warranted to infer ∀xA under the premiss that A(y/x) has been derived for an *arbitrary* object. The proviso (*) is intended to grant that the entity assigned to y in A(y/x) is wholly arbitrary, since it requires that y is 'fresh', not occurring in any of the assumptions that A(y/x) depends on. Let us see through an example how violating the proviso an invalid formula can be derived:

(23)
$$\cfrac{\cfrac{[P(y)]^1}{\forall xP(x)}\,\forall\mathbf{I}}{P(y)\supset\forall xP(x)}\supset\mathbf{I},1$$

The application of **∀I** violates the restriction, since the premiss P(y) depends on the assumption P(y), as we have seen, and of course y occurs in it. On the other hand, the conclusion is clearly invalid: according to it, if y is a man, then everything is a man.
- Proviso (**). The rule **∃E** can justified by the following reasoning. Suppose we have derived ∃xA(x), i.e. that the class of entities for which A holds is nonempty; assume A(y), i.e. call "y" an *arbitrary* element of that class (of course y must not occurr free in ∃xA, as (**) requires, for otherwise it might happen that we attribute to its denotation properties it has not); if, on the basis of this assumption, we can derive a formula B not containing y and not depending on assumptions containing y (as (**) requires), then we have derived B independently of A(y). Here is an example of how, violating the proviso, an invalid formula can be derived:

(24)
$$\cfrac{\cfrac{\cfrac{[\exists xQ(x)]^4 \qquad \cfrac{[\exists xP(x)]^3 \qquad \cfrac{\cfrac{[P(y)]^1 \qquad [Q(y)]^2}{P(y)\wedge Q(y)}\,\wedge\mathbf{I}}{\exists x(P(x)\wedge Q(x))}\,\exists\mathbf{I}}{\exists x(P(x)\wedge Q(x))}\,\exists\mathbf{E},1}{\exists x(P(x)\wedge Q(x))}\,\exists\mathbf{E},2}{\exists xQ(x)\supset\exists x(P(x)\wedge Q(x))}\supset\mathbf{I},4}{\exists xP(x)\supset(\exists xQ(x)\supset\exists x(P(x)\wedge Q(x)))}\supset\mathbf{I},3$$

The first application of **∃E** violates the proviso: ∃x(P(x)∧Q(x)) is B and P(y) is the assumption A(y/x); therefore there is an assumption on which B depends, i.e. Q(y), in which y occurs. On the other hand, the conclusion is clearly invalid: according to it, if some man is brown-haired and some man is blond-haired, then some man is brown-haired and blond-haired.

Let us conclude with some examples of arguments in natural deduction. Consider *Peirce's Law*: ((A⊃B)⊃A)⊃A; it is classically valid (show it), but not intuitionistically valid (show it); but intuitionisticslly it is possible to prove a slightly weaker theorem: ((A⊃B)⊃A)⊃¬¬A; here is a proof:

(25)

[(A⊃B)⊃A]   [(A⊃B)]

———————————

A   [¬A]

——————————————————

⊥

——————————————————

(A⊃B)⊃⊥

——————————————

¬¬A

——————————————

((A⊃B)⊃A)⊃¬¬A

Other examples:

Insert Troelstra & van Dalen, Constructivism in Mathematics, vol. 1, North-Holland, 1988, pp. 42-47.

We will use the symbol "$\Gamma\vdash_{IPrL}p$" to say that there is a derivation of p from the (open) assumptions in Γ; when the set Γ is empty (i.e., all assumptions from which p has been derived are discharged), we write $\vdash_{IPrL}p$, and we say that p is a(n IPrL) *theorem*.

2.4. The relation of semantics and proofs.

As in the case of classical propositional logic CPrL, we can ask questions about how Kripke semantics and Gentzen natural deduction system are related. In fact, the following facts can be proved:

(i) **Soundness (meta-)theorem**: If $\Gamma\vdash_{IPrL}p$, then $\Gamma\vDash_{IPrL}p$

(ii) **Completeness (meta-)theorem**: If $\Gamma\vDash_{IPrL}p$, then $\Gamma\vdash_{IPrL}p$

3.Validity and Logical Validity

3.1. Why Tarski's approach is unsatisfying

Let us come back to our starting question: what *is* a logically valid argument? We will illustrate now the intuitionistic answer. But a premise is necessary. As a matter of fact, we have *already* defined the notion of intuitionistic validity of arguments formalized both in IPL and in IPrL, so wqhy am I stating again the question? It is important to realize that the notions we have defined so far can be accepted and used by intuitionists as *metamathematical tools*, but not as *intended* notions of validity.

First: they can be accepted and used by intuitionists as *metamathematical tools*. Let me give an example of their use as metamathematical tools. Suppose we want to know whether a sentence p is a logical consequence of the set Γ of premises. We try to construct a derivation of p from the premises; if p *is* a logical consequence of Γ, such a derivation exists, and sooner or later we will find it; but if p *is not* a logical consequence of Γ, such a derivation does not exists: in this case, we will not find it after any finite time, but we will not be certain that it does not exist (we might find it tomorrow). In order to be certain that the derivation does not exist we must use another method: we construct a *countermodel* of the argument, i.e. a model in which all the premises in Γ are true and p is false; since we know (from the soundness metatheorem) that if there is a countermodel of an argument then the argument is not syntactically valid, we can conclude that a derivation from Γ to p does not exist. We have used the formal semantics given above for IPC and IPrC (Kripke semantics) as a metamathematical tool; but this is far from entailing that such formal semantics is 'good' from other standpoints, if we are intuitionists.

Second: they are not the *intended* notions of validity. There are several reasons for this claim, but I will restrict myself to one. The notions of validity and logical validity we have defined so far make an essential reference to the notion of classical truth: an argument is valid if it is truth-preserving. They are implementations of the classical, Tarskian, view of logical consequence, summarized in Definition 3. However, it is legitimate to be suspicious of Tarski's claim. John Etchemendy has raised, among many others, the following objection against this approach:[14]

> A logically valid argument must, at the very least, be capable of justifying its conclusion. It must be possible to come to know that the conclusion is true on the basis of knowledge that the argument is valid and that its premises are true. This is a feature of logically valid arguments that even those most skeptical of modal notions recognize as essential. Now, if we equate logical validity with mere truth preservation, [...], we obviously miss the essential characteristic of validity. For in general, it will be impossible to know both that an argument is "valid" (in this sense) and that its premises are true, without *antecendently* knowing that the conclusion is true.

Dag Prawitz makes essentially the same point:[15]

> The validity of inferences is not a small theme of philosophy. It is said that with the help of valid inferences, we justify our beliefs and acquire knowledge. The modal character of a valid inference is essential here, and is commonly articulated by saying that a valid inference guarantees the truth of the conclusion, given the truth of the premisses. It is because of this guarantee that a belief in the truth of the conclusion becomes justified when it has been inferred by the use of a valid inference from premisses known to be true. But if the validity of an inference is equated with (1) (or its variants), then in order to know that the inference is valid, we must already know, it seems, that the conclusion is true in case the premisses are true. After all, according to this analysis, the validity of the inference just means that the conclusion is true in case the premisses are, and that the same relation holds for all

[14] Etchemendy (1990), p. 93.

[15] Prawitz (2005), p. 675.

> inferences of the same logical form as the given one. Hence, on this view, we cannot really say that we infer the truth of the conclusion by the use of a valid inference. It is, rather, the other way around: we can conclude that the inference is valid after having established for all inferences of the same form that the conclusion is true in all cases where the premisses are.
>
> To say that a valid inference guarantees the truth of the conclusion, given the truth of the premisses, is to give the modal character of an inference an epistemic ring, and it seems obvious that [Definition 3] has no connection with such a modality.

Let me try to make the argument explicit:

(i) For every subject s, a valid inference **I** from A to B is intuitively useful for s if, and only if, for every time t, s knows at t that B only if there is a time t' such that (i) t'<t; (ii) s knows at t' that **I** is valid; (iii) s knows at t' that A ; (iii) s does not know at t' that B.

(ii) A good explanation of the validity of an inference **I** must account for its utility, i.e. explain how **I** can be at the same time valid and useful.

(iii) Assume that we define an inference as valid if, and only if, it preserves (realistic) truth.

(iv) From (iii) and (i) it follows that, for every subject s, a truth-preserving inference **I** is useful for s if, and only if, for every time t, s knows at t that B only if there is a time t' such that (i) t'<t; (ii) s knows at t' that **I** is truth-preserving; (iii) s knows at t' that A ; (iv) s does not know at t' that B.

(v) Condition (iv)(ii) means, by Tarski's definition of truth-preserving inference, that, for every model M, either s knows at t' that A is false in M or s knows at t' that B is true in M.

(vi) Then there cannot be a time t' satisfying the conditions (i)-(iv) specified in (iv); for, if (ii) holds, then, by (v), for every model M, either s knows at t' that A is false in M or s knows at t' that B is true in M; by (iii), s does not know at t' that A is false in M; hence s knows at t' that B is true in M, in contradiction with (iv).

(vii) Hence, if we equate the validity of an inference with its being truth-preserving, a valid inference cannot be useful; by (ii), the definition of validity as truth preservation does not account for its utility.

In the terms I have introduced during the classes, we could say that Tarski's definition of validity is not apt to suggest a *criterion* of validity of an argument, i.e. a way to *recognize* the validity of that argument.

A clarification is worthwhile here. What I have just said seems to be contradicted by the fact that, for many formal systems S (in particular for all the formal systems we have studied), we do have a criterion of validity: as S is complete, if $\Gamma\vDash_S p$ then we can generate an S-proof of it from Γ; and as S is sound, if not $\Gamma\vDash_S p$ we can generate an S-model of Γ in which p is false; assuming that the *intuitive* relation of logical consequence (which could be expressed by "$\Gamma\vDash p$") is adequately represented by the formal relation of S-logical consequence $\Gamma\vDash_S p$, we have a way to recognize the intuitive validity of an argument.
If fact there is no contradiction. In principle, it is perfectly possibile that, for some formal system S, S is not sound or not complete; in such a case the criterion of validity described cannot be applied. Moreover, there are mathematical formal theories T (for instance PA, Peano Arithmetic) that are incomplete in the sense that there are sentences p belonging to their language that are true in the intended model $\mathcal{N}$ of T (i.e. $\vDash_{\mathcal{N}} p$) but such that neither $\vdash_T p$ nor $\vdash_T \neg p$; since truth in the intended

model can be identified with intuitive truth, in such theories the intuitive relation $\Gamma \vDash p$ cannot be assumed to be adequately represented by the formal relation of S-logical consequence.

We present now the intuitionistic answer, essentially due to Prawitz himself.[16] The problem, as we have seen, is to understand the modal character of valid inference in epistemic terms. Here is how Prawitz articulates his proposal:[17]

> [...] I spoke of the truth of the premisses guaranteeing the truth of the conclusion. Another way of bringing out an epistemic force of necessity more clearly is to say
>
> [(18)] The truth of A follows by necessity of thought from the truth of all the sentences of Γ.
>
> In the same direction there are formulations such as one is committed to holding A true, having accepted the truth of the sentences of Γ; one is compelled to hold A true, given that one holds all the sentences of Γ true; on pain of irrationality, one must accept the truth of A, having accepted the truth of the sentences of Γ.
>
> To develop the idea of a necessity of thought more clearly we must bring in reasoning or proofs in some way. It must be because of an awareness of a piece of reasoning or a proof that one gets compelled to hold A true given that one holds all the sentences of Γ true.
>
> Let us call a verbalized piece of reasoning an *argument*, and let us speak of an argument *for A from* Γ, if Γ is the set of hypotheses on which the reasoning depends and A is the conclusion of the reasoning. By a *proof* of A from Γ we may understand either a valid argument for A from Γ or, more abstractly, what such an argument represents; I shall here reserve it for the latter use.
>
> A direction in which we may try to explicate [(18)] may then be put in one of the following two ways:
> [(19)] There is a valid argument for A from Γ; there is a proof of A from Γ.
> It may be said that condition [(19)] does not in itself involve any idea of necessity of thought, since the mere existence of a valid argument or of a proof does not compel us to anything. Condition [(19)] may nevertheless be said to bring in the idea of a necessity of thought by requiring the existence of something such that when we know of it, we are compelled to hold A true, given that we hold the sentences of Γ true. [...]
>
> We have thus to account for what it is that makes something a proof or, alternatively, to account for the validity of arguments. It should be noted at this point that not infrequently the validity of an argument is defined in terms of logical consequence. The idea here is to reverse the order of explanation in an attempt to catch a modal element in logical consequence. In other words, the validity of an argument for A from Γ is to be taken as a more basic notion, and is to be analyzed so that it constitutes evidence for A when given Γ, i.e., something that by necessity of thought makes us conclude A, given Γ.
>
> What is it that makes an argument valid and thus compels us, by necessity of thought, to hold the conclusion true, given the truth of the premisses? It is difficult to think of any answer that does not bring in the meaning of the sentences in question. In the end it must be because of the meaning of the expressions involved that we get committed to holding one sentence true, given the truth of some other sentences.
>
> To get further, we should thus turn to the notion of meaning.

3.2. Meaning

You have seen in the first Module the intuitionistic explanation of the meaning of the logical constants (the so-called BHK-explanation). I only remind that meaning is explained in terms of *proof*. But what is a proof, exactly? Heyting's definition of the notion "proof of A" by induction on the logical complexity of A is one answer; the leading idea of Heyting's definition is that a proof of A is, in mathematics, what confers *evidence* to sentences; we can therefore generalize Heyting's view of meaning by saying that the key-notion in terms of which meaning is explained is (not the classical notion of truth, but) the notion of evidence. But there is another kind of answer to our

[16] Cp. Prawitz (1985) and (2005).
[17] Prawitz (2005), pp. 677-678.

question: a proof is a sequence of *inferential steps*, each of which transmits evidence from the premises to the conclusion; also in this view proofs transmit evidence, but attention is paid to the linguistic presentations of proofs, i.e. arguments built up by means of inferential steps, as happens in natural deduction derivations. This second notion of proof was privileged by Gentzen, and later by Dummett, Prawitz and others, who gave rise to a tradition known today under the name of "neo-verificationism".

Although Gentzen's main purpose in Gentzen (1935) was to give a formalization of logic (the proof system illustrated above under the name Natural Deduction), when he explains the rules of his formal system he suggests the basic idea of a constructive theory of meaning, under some respects similar to Heyting's theory of meaning, under other respects sensibly different. Here is how he states it:

> The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact can be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only 'in the sense afforded it by the introduction of that symbol'. An example may clarify what is meant: We were able to introduce the formula A⊃B when there existed a derivation of B from the assumption formula A. If we then wished to use that formula by eliminating the ⊃-symbol [...], we could do this precisely by inferring B directly, once A has been proved, for what A⊃B attests is just the existence of a derivation of B from A. [...] By making these ideas more precise it should be possible to display the E-inferences as unique functions of their corresponding I-inferences, on the basis of certain requirements. [18]

An idea analogous under some respects is present in Wittgenstein's writings posterior to the *Tractatus*. For example he writes in Wittgenstein (1953):

> We can conceive the rules of inference – I want to say – as giving the signs their meaning, because they are rules for the use of these signs. So that the rules of inference are involved in the determination of the meaning of the signs. In this sense rules of inference cannot be right or wrong. [19]

The analogy may be expressed by the slogan "meaning is use": the meaning of every expression of a language is determined by the rules for its correct use; in particular, the meaning of the logical constants is determined by the role they play in the activity of inferring conclusions from premisses.

However, an important difference between Wittgenstein and Gentzen emerges from the quoted passages: Wittgenstein's idea that rules of inference cannot be right or wrong is totally absent from Gentzen; on the contrary, Gentzen's passage suggests a clear way in which an E-rule may be wrong: precisely when it does not satisfy the condition imposed by the corresponding I-rule. We will see in a moment an example of this possibility.

It is interesting to observe that Heyting's theory of meaning is different both from Wittgenstein's and from Gentzen's in that the idea that either the totality or a subset of the rules for the use of an expression (a logical constant) determine its meaning is absent from it; in other terms, the idea encoded in the slogan «Meaning is use» is extraneous to intuitionism: Heyting characterizes the intuitionistic meaning of the logical constants via his inductive definition of the notion "proof of A", not via rules or axioms. This does not at all prevent a system of rules from being adequate to the intuitionistic definition.

Wittgenstein's and Gentzen's ideas have given rise to two different – although related – approaches to the characterization of meaning in general, and in particular of the meaning of the logical constant: *inferentialism* (or *conceptual role semantics*: W. Sellars, G. Harman, H. Field, etc.), on the one hand, and *neo-verificationism* (or, more recently, *proof-theoretical semantics*:

---

[18] Gentzen (1935), pp. 80-1.

[19] Wittgenstein (1953), V. 23.

Dummett, Prawitz, Martin-Löf, etc.), on the other. Here we shall be concerned with neo-verificationism.

To sum up, the basic ideas the neoverificationists borrow from Gentzen are the following:

(i) Proofs are given to us through language, in the form of derivations or *arguments*;
(ii) The most 'natural' and significant analysis of deductive practice in mathematics, and therefore of deductive arguments, is Gentzen's systems of Natural Deduction, in which inference rules are seen as rules for the (correct) *use* of each logical constant;
(iii) For each logical constant, we must distinguish *two kinds* of its use: a kind of use that is *constitutive* of its meaning (exemplified by introduction rules); and a kind of use that is a consequence of the meaning of the constant established by the first kind.[20]

3.3. Two problems, and two neoverificationist responses.
However, for these ideas to become a true program for a theory of meaning of the logical constants, some basic difficulties must be tackled.

3.3.1. Tonk
The first is vividly stated by A. N. Prior in Prior (1960). Let us add to our language the constant TONK and define its meaning by the following rules:

$$\text{TONK-}\mathbf{I}\text{:}\quad \frac{A}{A\,\text{TONK}\,B} \qquad\qquad \text{TONK-}\mathbf{E}\text{:}\quad \frac{A\,\text{TONK}\,B}{B}$$

By the transitivity of the relation of deducibility it follows that $A \vdash B$, for every A and B - a rather uncomfortable outcome against which we cannot legitimately oppose that such a constant doesn't exist if we have abandoned «the old superstitious view that an expression must have some independently determined meaning before we can discover whether inferences involving it are valid or invalid.»[21]

Insert J. MacFarlane, "Logical Consequence"
online: McF logical-consequence.pdf

3.3.2. Uniqueness
The second difficulty is stated again by Prior, in Prior (1964). It is possible - he observes - that *several* distinct propositions correspond to one and the same definition. For instance, in a language rich enough for the formulation of the propositional calculus there is actually an infinite number of non-synonymous connectives with the same truth table. Let us suppose we have defined (in some way) the meaning of $\wedge$ and $\neg$; then we can define the following infinite sequence of 'conjunction-

[20] As we have seen above, it is necessary to make a clear distinction between the *intuitive* relation of logical consequence and the relation of logical consequence *formalized* by some logical system. As a concequence it is also necessary to make a clear distinction between *intuitive* proofs and proofs *formalized* by some logical system. When the neoverificationists explain the meaning of the logical constants in terms of the notion of proof, they are referring to *intuitive* proofs, even though they do assume that rules are of two kinds, and that introduction rules are meaning-constitutive. So we can say that they are referring to a specific set of introduction rules, but not to a specific set of rules of other kinds (eliminations or whatever).

proofs in terms of which the meaning of the logical constants is explained and the proofs of a formal deductive system.
[21] Prior (1960), pp. 129-30.

forming connectives':

$A\wedge_0 B \quad = \quad A\wedge B$
$A\wedge_{n+1} B \quad = \quad \neg(A\wedge_n\neg B)\wedge\neg(\neg A\wedge_n B)\wedge\neg(\neg A\wedge_n\neg B).$

All these connectives clearly have the same truth table, but they can hardly be synonymous.

MacFarlane ("Logical Consequence", p. 4) reminds that Belnap proposes a proof-theoretic analogue of uniqueness: to say that *plonk* describes a unique connective is to say that if another connective *plink* is given the same introduction and elimination rules, then they are proof-theoretically equivalent. If we take this as meaning that *plink* and *plonk* are synonymous, Belnap's proposal amounts to taking proof-theoretical (or, more generally, logical) equivalence as a criterion of synonymy. Consequently, Prior's objection is a critique to Belnap's proposal: the condition of logical equivalence is, according to him, a necessary, but not sufficient, condition of synonymy, since the connectives of the sequence above meet the condition, but considering them synonymous appears to be quite illegitimate: «I cannot see - Prior remarks - how the sense of a sentence can ever be identical with a logical complication of itself.»[22] Concerning this point no step forward seems to have been taken since Belnap's paper; Prior's objection remains unanswered.

It is interesting to point out that Heyting's theory of meaning does not incur the objection. $\neg(A\vee B)$ is logically equivalent (even intuitionistically) to $\neg A\wedge\neg B$, so the two sentences are synonymous, i.e. express the same proposition, according to Belnap's proposal; but a proof of $\neg(A\vee B)$ is *not* a proof of $\neg A\wedge\neg B$, according to the BHK-explanation; and what individuates the proposition expressed by a sentence, in Heyting's theory of meaning, is precisely the class of its proofs; as a consequence the two sentences would not be synonymous, according to Heyting.

3.3.3. Prawitz's definition of valid argument.

We can now explain the neoverificationist, anti-realist, account of consequence proposed by Prawitz's as an alternative to the realist, tarskian, one illustrated in Module 1.
At the same time, valid arguments are conceived by Prawitz as linguistic presentations of intuitionistic proofs (i.e. of proofs as defined by Heyting).

Insert J. MacFarlane, "Prawitz's proof-theoretic account of consequence"
online: McF prawitz.pdf

SOME NOTES TO BE ADDED TO MACFARLANE'S PAPER.

A. Reduction and justification.

As MacFarlane shows, when we show that an elimination rule is valid we show, in fact, that it is dispensible, and this dispensibility plays the role of a *justification* of the rule. For each elimination rule E, the key passage of such a justification consists in showing how a non-canonical argument terminating with E can be transformed into, or better *reduced to*, a canonical one. For example, in the case of $\wedge$**E** we justify the rule by performing the following reduction:

$$(26)\quad \dfrac{\dfrac{\begin{matrix}\Delta_4 & \Delta_5\\ A & B\end{matrix}}{A\wedge B}}{A} \quad \text{reduces to} \quad \begin{matrix}\Delta_4\\ A\end{matrix}$$

[22] Prior (1964), p. 193.

In the case of ∨**E** the rule is justified by performing the following reduction (I consider only one possibility, the other is analogous):

(27)
$$\frac{\begin{matrix}\Delta_3 & [A] & [B] \\ A & \Delta_1 & \Delta_2 \\ A\vee B & C & C\end{matrix}}{C} \qquad \text{reduces to} \qquad \begin{matrix}A \\ \Delta_1 \\ C\end{matrix}$$

In fact there are justifying reductions for all elimination rules of Gentzen's system of natural deduction.[23]

But several other kinds of rules can be justified by appropriate reductions. In general, therefore, reductions can be seen as *justifying operations*, and Prawitz defines a notion of valid argument relativized to a given set of justifying operations. In order to stress this relativization it is convenient to modify the definition of validity given by MacFarlane in the following way:

(28) **Definition**

An argument is *J-canonical* iff it ends with an introduction rule and contains J-valid (i.e. valid relative to a set J of justifying operations)(open or closed) arguments for the premises.

- A *closed* argument is *J-valid* iff either (a) it is a J-canonical argument or (b) the operations in J provide an effective way of reducing it to a J-canonical argument for its conclusion.

- An *open* argument is *J-valid* iff, for every extension J′ of J, the result of replacing each assumption with a J′-valid closed argument for that assumption is J′-valid.[24]

B. The role of canonical arguments

We have seen that

- according to intuitionists, an understanding of a mathematical statement consists in the capacity to recognize a proof of it when presented with one;
- Heyting explained what is a proof of A by means of his inductive characterization;
- according to neoverificationists, proofs are given to us through language, in the form of arguments; more precisely, a proof of A is given, or presented, by a valid closed argument having A as conclusion.

Let's call the notion in terms of which the meaning of the sentences of a language is explained the *key-notion* of the theory of meaning for that language. In this terminology *proof* is the key-notion of the intuitionistic theory of meaning; can we say that *valid argument* is the key-notion of the neoverificationist theory of meaning?

Not exactly. As we have seen, for Gentzen and the neoverificationists it is essential to distinguish between two kinds of rules, and consequently between canonical and non-canonical arguments, while this distinction is *not* present in the intuitionistic theory of meaning. Neoverificationists find this absence 'misleading' or inaccurate, but in any case necessary even within the intuitionistic framework[25]. This thesis is questionable, but it clearly suggests that the key-notion of the neoverificationist theory of meaning should be the notion of *canonical argument*,

---

[23] These reductions are used in the proof of one of the basic theorems of proof theory, the Normalization Theorem.

[24] The reason of the reference, in the last clause, to extensions of J is rather involved, and may be skipped. Here it is. If there were not the restriction, there might be some open argument that is J-valid but not J′-valid, for some extension J′ of J. This would happen in the following situation. Suppose we have an open argument π with premiss A and conclusion B that is J-valid; suppose that J′ contains operations justifying inferences not justified by J, and that these inferences provide J′-canonical arguments for A that cannot be transformed into J′-canonical arguments for B; in this situation, π is not J′-valid; in particular the implication A⊃B, which was J-valid, is not J′-valid: and this would conflict with the strong intuition that validity is a *stable* property, in the sense that if A⊃B is valid at stage t it is still valid at every later stage t′.

[25] Dummett (1977), pp. 392 ff..

since the only rules that are constitutive of meaning are introductions, and canonical arguments are defined in terms of introduction rules.

Bibliographical References

Dummett, Michael (1977) *Elements of Intuitionism*, Clarendon Press, Oxford.

Etchemendy, John (1990) *The Concept of Logical Consequence*, Cambridge (Mass.)-London, Harvard University Press.

Gentzen, Gerhard (1935) "Untersuchungen über das logische Schliessen", *Mathematische Zeitschrift,* 39, pp. 176-210, 405-431. [Engl. transl. "Investigations into Logical Deduction", in *Collected Papers*, ed. by M. E. Szabo, Amsterdam, North Holland, 1969, pp. 68-131.]

Prawitz, Dag (2005) "Logical Consequence from a Constructivist Point of View", in S. Shapiro (ed.), *The Oxford Handbook of Philosophy of Mathematics and Logic*, Oxford University Press, pp. 671-695.

Prior, Arthur N. (1960) "The runabout inference ticket" *Analysis* 21; pp.38-9.

Prior, Arthur N. (1964), "Conjunction and contonktion revisited", *Analysis*, 24, pp. 191-195.

Tarski, Alfred (1935) "On the Concept of Logical Consequence", in *Logic, Semantics, Metamathematics*, Oxford, Clarendon Press, 1956, pp. 409-420.